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The Stochastic Law of the Busy Period for a Single Server Queue with Poisson Input

LAJOS TAKÁCS

Columbia University, New York, New York

Submitted by Richard Bellman

I. INTRODUCTION

An interesting problem in the theory of queues is to find the stochastic law of the busy period for a single server queue. Suppose that customers arrive at a counter at times $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ where the interarrival times $\tau_{n+1} - \tau_n$ $(n = 1, 2, \cdots)$ are identically distributed, mutually independent random variables with distribution function

$$\mathbf{P}\{\tau_{n+1} - \tau_n \leqq x\} = F(x) \qquad (n = 1, 2, \cdots).$$

The input is said to be a recurrent process. The customers are served by a single server. The server is idle if and only if there is no customer in the system. Denote by χ_n the service time of the nth arriving customer. It is supposed that $\{\chi_n\}$ is a sequence of identically distributed, mutually independent, positive random variables with distribution function

$$\mathbf{P}\{\chi_n \leqq x\} = H(x) \qquad (n = 1, 2, \cdots)$$

and independent of the input process. The busy period is defined as the time interval during which the server is continuously busy. Busy periods and idle periods alternate. Evidently every busy period independently of the others has the same stochastic law. Denote by $G_n^*(x)$ the probability that a busy period consists of n services and has length $\leqq x$. Define

$$\Gamma_n^*(s) = \int_0^\infty e^{-sx}\, dG_n^*(x) \tag{1}$$

for $\Re(s) \geqq 0$ and

$$\Gamma(s, w) = \sum_{n=1}^{\infty} \Gamma_n^*(s)\, w^n \tag{2}$$

for $\Re(s) \geqq 0$ and $|w| \leqq 1$.

Evidently $G_n^*(\infty)$ is the probability that a busy period consists of n services and

$$G(x) = \sum_{n=1}^{\infty} G_n^*(x) \tag{3}$$

is the probability that a busy period has length $\leqq x$. Let

$$\Gamma(s) = \int_0^\infty e^{-sx}\, dG(x) \tag{4}$$

for $\Re(s) \geqq 0$.

Several authors investigated the stochastic law of the busy period for different queueing processes. In 1942 Borel [1] found $G_n^*(x)$ for Poisson input and constant service times. In 1951 Kendall [2] derived a functional equation for $\Gamma(s)$ in the case of Poisson input and general service times. In 1952 Pollaczek [3] gave a complex integral expression for $\Gamma(s, w)$ in the case of recurrent input and general service times. During the last ten years several methods have been used to investigate the stochastic law of the busy period for different queueing processes. The method of functional equations was used by the author [4, 5]. Rouché's theorem was used by Gaver [6], and the author [7-9, 5]. The technique of difference equations was introduced by Conolly [10, 11]. Mathematical induction was applied by Prabhu [12]. Combinatorial methods were used by Tanner [13] and the author [14-17]. The method of integral equations was introduced by Rice [18]. Other problems concerning busy periods were investigated by Gani [19], Gani and Prabhu [20], Gani and Pyke [21], Karlin and McGregor [22], Karlin *et al.* [23], McMillan and Riordan [24], Riordan [25], and Tanner [26].

The aim of this paper is to find the stochastic law of the busy period for a single server queue with Poisson input and general service times. The results are more complete than the earlier ones in the respect that we find not only the explicit form of $G_n^*(x)$ but the probability that the busy period has finite length or consists of a finite number of services.

II. An Auxiliary Theorem

Let $H(x)$ be the distribution function of a nonnegative random variable. The trivial case $H(x) = 1$ if $x \geqq 0$, $H(x) = 0$ if $x < 0$ is excluded. Define

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x) \tag{5}$$

for $\Re(s) \geqq 0$. The Laplace-Stieltjes transform $\psi(s)$ is a regular function of s

in the domain $\Re(s) > 0$ and is a continuous function of s in the domain $\Re(s) \geqq 0$. Always $|\psi(s)| \leqq 1$ if $\Re(s) \geqq 0$, $|\psi(s)| < 1$ if $\Re(s) > 0$ and $\psi(0) = 1$. Let

$$\alpha = \int_0^\infty x\, dH(x). \tag{6}$$

If α is finite then $\alpha = -\psi'(+0)$. Now we shall prove the following

LEMMA. *If $\lambda > 0$, $\Re(s) \geqq 0$ and $|w| \leqq 1$ then $z = \gamma(s, w)$, the root of the equation*

$$z = w\psi(s + \lambda(1 - z)) \tag{7}$$

that has the smallest absolute value, is

$$\gamma(s, w) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} w^n}{n!} \int_0^\infty e^{-(\lambda+s)x}\, x^{n-1}\, dH_n(x) \tag{8}$$

where $H_n(x)$ denotes the n-th iterated convolution of $H(x)$ with itself. Always $|\gamma(s, w)| \leqq 1$. If $\lambda\alpha \leqq 1$ then $\gamma(0, 1) = 1$ and if $\lambda\alpha > 1$ then $\gamma(0, 1)$ is real and $0 < \gamma(0, 1) < 1$. $\gamma(s, w)$ is a continuous function of s and w if $\Re(s) \geqq 0$, $|w| \leqq 1$ and is a regular function of s and w if $\Re(s) > 0$ and $|w| < 1$.

PROOF. Define

$$\gamma^*(s, w) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} w^n}{n!} \int_0^\infty e^{-(\lambda+s)x}\, x^{n-1}\, dH_n(x) \tag{9}$$

for $\Re(s) \geqq 0$ and $|w| \leqq 1$. We shall prove that $\gamma^*(s, w)$ is a continuous function of s and w if $\Re(s) \geqq 0$ and $|w| \leqq 1$ and it is a regular function of s and w if $\Re(s) > 0$ and $|w| < 1$. It is sufficient to prove that the series (9) is uniformly convergent in the domain $\Re(s) \geqq 0$, $|w| \leqq 1$ because the nth term of the series (9) is continuous in s and w if $\Re(s) \geqq 0$, $|w| \leqq 1$ and regular if $\Re(s) > 0$ and $|w| < 1$. If $\Re(s) \geqq 0$ and $x \geqq 0$ then

$$|e^{-(\lambda+s)x} (\lambda x)^{n-1}| \leqq e^{-\lambda x} (\lambda x)^{n-1} \leqq e^{-(n-1)} (n-1)^{n-1} \qquad (n = 1, 2, \cdots)$$

whence

$$\left| \frac{\lambda^{n-1} w^n}{n!} \int_0^\infty e^{-(\lambda+s)x}\, x^{n-1}\, dH_n(x) \right| \leqq \frac{e^{-(n-1)} (n-1)^{n-1}}{n!}$$

and the uniform convergence of (9) follows from Euler's formula

$$\sum_{n=1}^{\infty} \frac{e^{-(n-1)} n^{n-1}}{n!} = e. \tag{10}$$

We note that (7) has at most one root on the periphery of the unit circle; namely, $z = 1$ is a root if $w\psi(s) = 1$. If $z \neq 1$ and $|z| = 1$, then $|\psi(s + \lambda(1 - z))| < 1$ and, therefore, z cannot satisfy (7).

1. Let $\lambda\alpha > 1$. Then $|\psi(s + \lambda(1 - z))| < \psi(\lambda\epsilon) < 1 - \epsilon$ if $\epsilon > 0$ is small enough. For $\psi(\lambda\epsilon)$ and $1 - \epsilon$ are equal at $\epsilon = 0$ and their right hand derivatives at $\epsilon = 0$ are $-\lambda\alpha$ and -1 respectively where $-\lambda\alpha < -1$. Consequently, $|w\psi(s + \lambda(1 - z))| < |z|$ if $|z| = 1 - \epsilon$ and $\epsilon > 0$ is small enough. Since both $w\psi(s + \lambda(1 - z))$ and z are regular functions of z in the domain $|z| \leqq 1 - \epsilon$, it follows by Rouché's theorem that (7) has one and only one root, $z = \gamma(s, w)$, in the domain $|z| \leqq 1 - \epsilon$. The explicit form of $\gamma(s, w)$ can be obtained by Lagrange's expansion. (Cf. [27, p. 132].) Thus we get that $\gamma(s, w) = \gamma^*(s, w)$ if $\Re(s) \geqq 0$ and $|w| \leqq 1$. Here $\gamma(0, 1)$ is a positive real number and $|\gamma(0, 1)| < 1$. This proves the theorem for $\lambda\alpha > 1$.

2. Let $\lambda\alpha \leqq 1$. First we prove that (7) cannot have two distinct roots in the unit circle $|z| \leqq 1$. If $\Re(s) \geqq 0$ and $|z| \leqq 1$, then we have

$$\left| \frac{\partial\psi(s + \lambda(1 - z))}{\partial z} \right| < \lambda\alpha \qquad \text{for} \qquad z \neq 1$$

and

$$\left| \frac{\partial\psi(s + \lambda(1 - z))}{\partial z} \right| \leqq \lambda\alpha \qquad \text{for} \qquad z = 1.$$

If we suppose that z_1 and z_2 are two distinct roots of (7) in the unit circle $|z| \leqq 1$, then we obtain that

$$\begin{aligned} |z_2 - z_1| &= |w| \, |\psi(s + \lambda(1 - z_2)) - \psi(s + \lambda(1 - z_1))| \\ &= |w| \left| \int_{z_1}^{z_2} \frac{\partial\psi(s + \lambda(1 - z))}{\partial z} \, dz \right| < \lambda\alpha \, |z_2 - z_1|. \end{aligned}$$

This contradiction proves the statement. We shall distinguish two cases:

(a) $|w\psi(s)| < 1$. (This contains the case $\Re(s) > 0$.) If $|z| \leqq 1$, then $|w\psi(s + \lambda(1 - z))| \leqq |w\psi(s)| < 1 - \epsilon$ for a sufficiently small $\epsilon > 0$. Consequently, $|w\psi(s + \lambda(1 - z))| < |z|$ if $|z| = 1 - \epsilon$ and $\epsilon > 0$ is small enough. Thus by Rouché's theorem we can conclude that (7) has one only one root in the circle $|z| \leqq 1 - \epsilon$ and the explicit form of $z = \gamma(s, w)$ can be obtained by Lagrange's expansion. This proves that $\gamma(s, w) = \gamma^*(s, w)$ if $|w\psi(s)| < 1$ (therefore, if $\Re(s) > 0$).

(b) $|w\psi(s)| = 1$. (Then $\Re(s) = 0$ and $|w| = 1$.) Let $\lim_{n\to\infty} s_n = s$ where $\Re(s) = 0$ and $\Re(s_n) > 0$. We shall prove that

$$\lim_{n\to\infty} \gamma(s_n, w) = \gamma^*(s, w), \qquad |\gamma^*(s, w)| \leqq 1 \qquad \text{and} \qquad z = \gamma^*(s, w)$$

is a root of (7). Thus $\gamma(s, w) = \gamma^*(s, w)$ must hold also if $\Re(s) = 0$. Since $\gamma(s_n, w) = \gamma^*(s_n, w)$, $|\gamma^*(s_n, w)| \leqq 1$ and $\gamma^*(s, w)$ is continuous in the domain $\Re(s) \geqq 0$, $|w| \leqq 1$, we have $\lim_{n\to\infty} \gamma(s_n, w) = \gamma^*(s, w)$ with $|\gamma^*(s, w)| \leqq 1$. Further, if $n \to \infty$ in

$$\gamma(s_n, w) = w\psi(s + \lambda(1 - \gamma(s_n, w)),$$

then we get

$$\gamma^*(s, w) = w\psi(s + \lambda(1 - \gamma^*(s, w)).$$

Since (7) has at most one root in the domain $|z| \leqq 1$, $\gamma(s, w) = \gamma^*(s, w)$ must hold also in the case $\Re(s) = 0$. Accordingly, $\gamma(s, w) = \gamma^*(s, w)$ if $\lambda\alpha \leqq 1$ and $\Re(s) = 0$. If $s = 0$ and $w = 1$ then $z = 1$ is a root of (7) and, therefore, $\gamma(0, 1) = 1$ if $\lambda\alpha \leqq 1$. This completes the proof of the lemma.

REMARK 1. If $|w\psi(s)| = 1$, then $z = \gamma(s, w)$ is the root with smallest absolute value in z of the equation

$$z = w\psi(s)\,\psi(\lambda(1 - z)). \tag{11}$$

If $|w\psi(s)| = 1$, then $|w| = 1$ and $|\psi(s)| = 1$. In

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x)$$

$|e^{-sx}| \leqq 1$ and to satisfy $|\psi(s)| = 1$ it is necessary that the function $H(x)$ be constant in every interval of x in which $e^{-sx} \neq \psi(s)$. In other words, $H(x)$ can increase only at points x for which $e^{-sx} = \psi(s)$. Hence, if $|\psi(s)| = 1$, then

$$\psi(s + \lambda(1 - z)) = \int_0^\infty e^{-[s+\lambda(1-z)]x}\, dH(x) = \psi(s)\,\psi(\lambda(1 - z)). \tag{12}$$

Putting (12) into (7) we obtain (11).

REMARK 2. By forming the Lagrange expansion of $[\gamma(s, w)]^r$ for $r = 1, 2, \cdots$ we obtain that

$$[\gamma(s, w)]^r = r \sum_{n=r}^{\infty} \frac{w^n \lambda^{n-r}}{n(n-r)!} \int_0^\infty e^{-(\lambda+s)x}\, x^{n-r}\, dH_n(x). \tag{13}$$

If we put $s = 0$ and $w = 1$ in (13) and suppose that $\lambda\alpha \leqq 1$, then we obtain the following interesting identity

$$r \sum_{n=r}^{\infty} \frac{\lambda^{n-r}}{n(n-r)!} \int_0^\infty e^{-\lambda x}\, x^{n-r}\, dH_n(x) = 1. \tag{14}$$

If, in particular,

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geqq \alpha, \\ 0 & \text{if} \quad x < \alpha \end{cases}$$

in (14), then we obtain that

$$\sum_{n=r}^{\infty} \frac{r}{n} e^{-\lambda \alpha n} \frac{(\lambda \alpha n)^{n-r}}{(n-r)!} = 1 \tag{15}$$

for $\lambda\alpha \leqq 1$ and $r = 1, 2, \cdots$.

III. The Probability Law of the Busy Period

Now let us suppose that $\{\tau_n\}$ is a Poisson process of density λ, i.e.,

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x < 0, \end{cases} \tag{16}$$

whereas $H(x)$ is arbitrary. Then $\Gamma_n^*(s)$, the Laplace-Stieltjes transform of $G_n^*(x)$, is given by

Theorem 1. *If* $\Re(s) \geqq 0$ *and* $|w| \leqq 1$, *then*

$$\sum_{n=1}^{\infty} \Gamma_n^*(s)\, w^n = \gamma(s, w) \tag{17}$$

where $\gamma(s, w)$ *is the root with smallest absolute value in* z *of the equation*

$$z = w\psi(s + \lambda(1 - z)). \tag{18}$$

Proof. Denote by $G_{n,k}^*(x)$ the probability that a busy period consists of at least n services, that at the end of the nth service k customers are in the system and that the total service time of the first n customers is $\leqq x$. Then $G_n^*(x) = G_{n,0}^*(x)$. It is easy to see that

$$G_{1,k}^*(x) = \int_0^x e^{-\lambda u} \frac{(\lambda u)^k}{k!}\, dH(u) \tag{19}$$

and for $n = 1, 2, \cdots$

$$G_{n+1,k}^*(x) = \sum_{r=1}^{k+1} \int_0^x G_{n,r}^*(x-u)\, e^{-\lambda u} \frac{(\lambda u)^{k-r+1}}{(k-r+1)!}\, dH(u). \tag{20}$$

Let

$$\Gamma^*_{n,k}(s) = \int_0^\infty e^{-sx}\, dG^*_{n,k}(x) \qquad (\Re(s) \geqq 0). \tag{21}$$

Then $\Gamma^*_n(s) = \Gamma^*_{n,0}(s)$. Forming the Laplace-Stieltjes transforms of (19) and (20), we obtain

$$\Gamma^*_{1,k}(s) = \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!}\, dH(x) \tag{22}$$

and

$$\Gamma^*_{n+1,k}(s) = \sum_{r=1}^{k+1} \Gamma^*_{n,r}(s) \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^{k-r+1}}{(k-r+1)!}\, dH(x). \tag{23}$$

If we introduce the generating function

$$U_n(s, z) = \sum_{k=0}^{\infty} \Gamma^*_{n,k}(s)\, z^k \tag{24}$$

for $|z| \leqq 1$ and $\Re(s) \geqq 0$, then we have

$$U_1(s, z) = \psi(s + \lambda(1 - z))$$

and

$$zU_{n+1}(s, z) = \psi(s + \lambda(1 - z))\, [U_n(s, z) - \Gamma^*_n(s)]$$

whence, for $|w| < 1$,

$$\sum_{n=1}^{\infty} U_n(s, z)\, w^n = \frac{w\psi(s + \lambda(1 - z)) \left[z - \sum_{n=1}^{\infty} \Gamma^*_n(s)\, w^n\right]}{z - w\psi(s + \lambda(1 - z))}. \tag{25}$$

The left-hand side of (25) is bounded if $\Re(s) \geqq 0$, $|z| \leqq 1$ and $|w| < 1$, because $|U_n(s, z)| \leqq 1$. In this domain the denominator of the right-hand side of (25) has one and only one root $z = \gamma(s, w)$ defined by (8) and, therefore, this must be also a root of the numerator. Thus

$$\sum_{n=1}^{\infty} \Gamma^*_n(s)\, w^n = \gamma(s, w) \tag{26}$$

if $\Re(s) \geqq 0$ and $|w| < 1$. Since the left-hand side of (26) is a continuous function of w if $|w| \leqq 1$ and $\Re(s) \geqq 0$, we obtain by continuity that (26) is also valid for $|w| \leqq 1$ and $\Re(s) \geqq 0$. This completes the proof of the theorem.

THEOREM 2. *We have*

$$G_n^*(x) = \frac{\lambda^{n-1}}{n!} \int_0^x e^{-\lambda u} u^{n-1} \, dH_n(u) \qquad (n = 1, 2, \cdots). \tag{27}$$

PROOF. By (17) and (18)

$$\Gamma_n^*(s) = \frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-(\lambda+s)x} x^{n-1} \, dH_n(x) \tag{28}$$

whence (27) follows immediately.

REMARK 3. The probability that in r different busy periods altogether n customers are served and that these r busy periods have total length $\leqq x$ is given by

$$\frac{r}{n} \int_0^x e^{-\lambda u} \frac{(\lambda u)^{n-r}}{(n-r)!} \, dH_n(u). \tag{29}$$

For, the Laplace-Stieltjes transform of the probability in question is equal to the coefficient of w^n in (13).

THEOREM 3. *The probability that the length of a busy period is* $\leqq x$ *is given by*

$$G(x) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^x e^{-\lambda u} u^{n-1} \, dH_n(u). \tag{30}$$

If $\lambda\alpha \leqq 1$, *then* $G(\infty) = 1$, *whereas if* $\lambda\alpha > 1$, *then* $G(\infty) = \omega < 1$ *where* $z = \omega$ *is the only root of the equation*

$$z = \psi(\lambda(1 - z)) \tag{31}$$

in the unit circle $|z| < 1$.

PROOF. If we add (27) for $n = 1, 2, \cdots$, then we obtain (30). The Laplace-Stieltjes transform of $G(x)$ is $\Gamma(s) = \gamma(s, 1)$, i.e., the root with smallest absolute value in z of the equation

$$z = \psi(s + \lambda(1 - z)). \tag{32}$$

Thus $G(\infty) = \Gamma(0) = \gamma(0, 1)$. If $\lambda\alpha \leqq 1$, then $\gamma(0, 1) = 1$, whereas if $\lambda\alpha \geqq 1$, then $0 < \gamma(0, 1) < 1$. Evidently, $\gamma(0, 1)$ is the probability that a busy period has finite length or consists of a finite number of services.

REMARK 4. The rth iterated convolution of $G(x)$ with itself is

$$G_r(x) = r\sum_{n=r}^{\infty}\frac{1}{n}\int_0^x e^{-\lambda u}\frac{(\lambda u)^{n-r}}{(n-r)!}\,dH_n(u) \qquad (r = 1, 2, \cdots). \tag{33}$$

For, the Laplace-Stieltjes transform of $G_r(x)$ is $[\Gamma(s)]^r = [\gamma(s, 1)]^r$.

REMARK 5. Let

$$\alpha_r = \int_0^{\infty} x^r dH(x) = (-1)^r \psi^{(r)}(+0) \qquad (r = 1, 2, \cdots) \tag{34}$$

and

$$\Theta_r = \int_0^{\infty} x^r dG(x) = (-1)^r \Gamma^{(r)}(+0) \qquad (r = 0, 1, \cdots). \tag{35}$$

Since $\Gamma(s) = \gamma(s, 1)$ defined by (8), we obtain that

$$\Theta_r = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}\int_0^{\infty} e^{-\lambda x}\, x^{n-1+r}\, dH_n(x) \qquad (r = 0, 1, \cdots), \tag{36}$$

provided that it is finite. If we take into consideration that $z = \Gamma(s)$ satisfies (32), then we can determine Θ_r $(r = 1, 2, \cdots)$ recursively. If we suppose that $\lambda\alpha_1 < 1$ and α_r is finite, then $\Theta_0, \Theta_1, \cdots, \Theta_r$ are also finite and $\Theta_0 = 1$,

$$\Theta_1 = \frac{\alpha_1}{1-\lambda\alpha_1}, \qquad \Theta_2 = \frac{\alpha_2}{(1-\lambda\alpha_1)^3},$$
$$\Theta_3 = \frac{\alpha_3}{(1-\lambda\alpha_1)^4} + \frac{3\lambda\alpha_2^2}{(1-\lambda\alpha_1)^5}, \qquad \cdots. \tag{37}$$

If $\lambda\alpha_1 \geqq 1$, then Θ_r $(r = 0, 1, \cdots)$ can be obtained in a similar way.

REFERENCES

1. BOREL, É. Sur l'emploi du théorème de Bernoulli pour faciliter le calcul d'un infinité de coefficients. Application au problème de l'attente à un guichet. *Compt. rend. acad. sci. Paris* 214, 452-456 (1942).
2. KENDALL, D. G. Some problems in the theory of queues. *J. Roy. Statist. Soc.* Ser. B, 13, 151-185 (1951).
3. POLLACZEK, F. Sur la répartition des périodes d'occupation ininterrompue d'un guichet. *Compt. rend. acad. sci. Paris* 234, 2042-2044 (1952).
4. TAKÁCS, LAJOS. Investigation of waiting time problems by reduction to Markov processes. *Acta Math. Acad. Sci. Hung.* 6, 101-129 (1955).
5. TAKÁCS, LAJOS. The transient behavior of a single server queueing process with a Poisson input. *Proc. Fourth Berkeley Symposium on Math. Statist. and Probability*, Vol. 2, pp. 535-567. Univ. of California Press, Berkeley and Los Angeles, 1961.
6. GAVER, DONALD P. Imbedded Markov chain analysis of a waiting-line process in continuous time. *Ann. Math. Statist.* 30, 698-720 (1959).

7. TAKÁCS, LAJOS. Transient behavior of single-server queuing processes with recurrent input and exponentially distributed service times. *Operations Research* **8**, 231-245 (1960).
8. TAKÁCS, LAJOS. Transient behavior of single-server queueing processes with Erlang input. *Trans. Am. Math. Soc.* **100**, 1-28 (1961).
9. TAKÁCS, LAJOS. The transient behavior of a single server queuing process with recurrent input and gamma service time. *Ann. Math. Statist.* **32**, 1286-1298 (1961).
10. CONOLLY, B. W. The busy period in relation to the queueing process GI/M/1. *Biometrika* **46**, 246-251 (1959).
11. CONOLLY, B.W. The busy period in relation to the single-server queueing system with general independent arrivals and Erlangian service-time. *J. Roy. Statist. Soc. Ser. B*, **22**, 89-96 (1960).
12. PRABHU, N. U. Some results for the queue with Poisson arrivals. *J. Roy. Statist. Soc. Ser. B*, **22**, 104-107 (1960).
13. TANNER, J. C. A derivation of the Borel distribution. *Biometrika* **48**, 222-224 (1961).
14 TAKÁCS, LAJOS. The probability law of the busy period for two types of queuing processes. *Operations Research* **9**, 402-407 (1961).
15. TAKÁCS, LAJOS. The time dependence of a single-server queue with Poisson input and general service times. *Ann. Math. Statist.* (submitted).
16. TAKÁCS, LAJOS. A combinatorial method in the theory of queues. *J. Soc. Ind. Appl. Math.* **10** (1962) to appear.
17. TAKÁCS, LAJOS. A single-server queue with recurrent input and exponentially distributed service times. *Operations Research* **10**, 395-399 (1962).
18. RICE, S. O. Single-server systems—II. Busy periods. *Bell System Tech. J.* **41**, 279-310 (1962).
19. GANI, J. Elementary methods for an occupancy problem of storage. *Math. Ann.* **136**, 454-465 (1958).
20. GANI, J. AND PRABHU, N. U. The time-dependent solution for a storage model with Poisson input. *J. Math. and Mech.* **8**, 653-663 (1959).
21. GANI, J. AND PYKE, R. The content of a dam as the supremum of an infinitely divisible process. *J. Math. and Mech.* **9**, 639-651 (1960).
22. KARLIN, SAMUEL AND MCGREGOR, JAMES. Many server queueing processes with Poisson input and exponential service times. *Pacific J. Math.* **8**, 87-118 (1958).
23. KARLIN, S., MILLER, R. G. AND PRABHU, N. U. Note on a moving single server problem. *Ann. Math. Statist.* **30**, 243-246 (1959).
24. MCMILLAN, B. AND RIORDAN, J. A moving single server problem. *Ann. Statist.* **28**, 471-478 (1957).
25. RIORDAN, JOHN. Delays for last-come first-served service and the busy period. *Bell System Tech. J.* **40**, 785-793 (1961).
26. TANNER, J. C. A problem of interference between two queues. *Biometrika* **40**, 58-69 (1953).
27. WHITTAKER, E. T. AND WATSON, G. N. "A Course of Modern Analysis." Cambridge Univ. Press, 1952.
28. POLLACZEK, FÉLIX. "Problèmes Stochastiques Posés par le Phénomène de Formation d'une Queue d'Attente à un Guichet et par des Phénomènes Apparentés." Gauthier-Villars, Paris, 1957.
29. TAKÁCS, LAJOS. "Introduction to the Theory of Queues." Oxford Univ. Press, New York, 1962.